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Water-wave transmission through barriers with small gaps

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SUMMARY

The problem of the transmission of water waves through gaps in n arbitrary barriers is solved under the assumption that both the gaps and the barrier thicknesses are small. Closed-form expressions for the transmission and reflection coefficients are derived for the special case of n equally-spaced identical barriers and gaps.

1. Introduction

We consider the two-dimensional problem of the transmission of infinitesimal-amplitude surface waves through small horizontal gaps in a series of n parallel vertical barriers of small thickness which extend indefinitely down into the fluid from the free surface. It is assumed that the gap in each barrier and the barrier thicknesses are of the same order of magnitude, each being small compared to the other length scales in the problem, namely the incident wave-length, the depth of submergence of the gaps, and the spacing of the barriers. The method used to solve the problem is to match inner solutions, valid in the neighbourhood of each of the gaps, and describing their detailed geometry, with outer solutions, valid at large distances from the gaps, and describing the wave-like character of the flow, in an intermediate region where both expansions are assumed to be valid. The method was used by Tuck [1] for the case of a small gap in a single thin barrier. This problem can be solved exactly, on linear theory, and Tuck's results were shown by Guiney [2] to agree with the exact result, for low frequencies, even when the size of the gap was up to twice the depth of the upper edge of the gap. For large frequencies the approximation became progressively worse as the gap size increased relative to the gap depth.

The effect of barrier thickness was considered by Guiney, Noye and Tuck [3] who showed that the amount of energy transmitted was less for thicker barriers. The only modification to the solution was that it was found necessary to use a Schwarz–Christoffel mapping to solve for the "inner solution" valid in the neighbourhood of the gap, which described the streaming flow through a finite rectangular aperture in a wall of finite thickness in an infinite fluid, having source-sink like behaviour at infinity. A method equivalent to, but not as general as the matching method to be used here, was used by the author in considering two thin barriers with symmetric small gaps. It was shown (Evans [4]) that there exist configurations for which total reflection of the incident wave occurred. These results will be confirmed as a special case of the present problem corresponding to $n = 2$, with thin barriers and symmetric gaps.

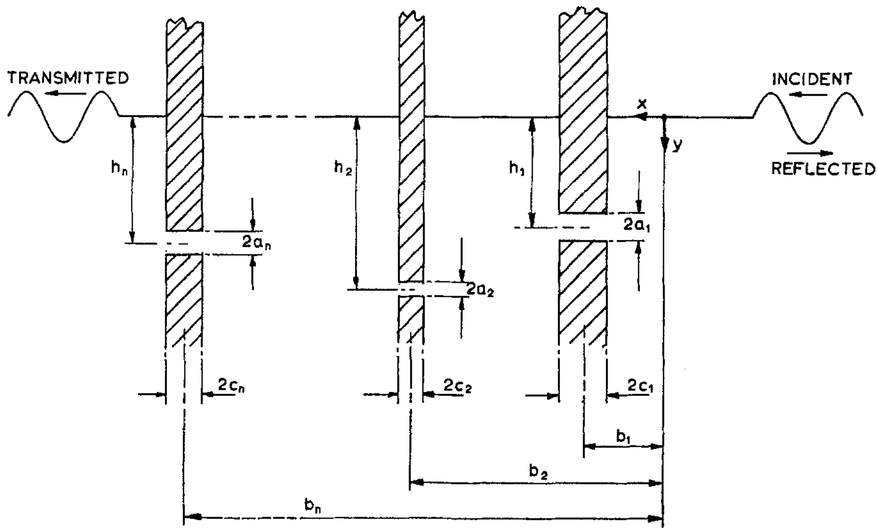

Figure 1.

2. Formulation

The geometry of the problem is sketched in Fig. 1. A surface wave, of frequency ω, is incident from $x = -\infty$ upon the barriers which occupy the positions $|x - b_r| \leqq c_r$ with the gaps L_r: $|y - h_r| \leqq a_r$, $r = 1, 2, \ldots, n$. The usual assumptions of linearised water-wave theory ensure the existence of a velocity potential $\Phi(x, y, t)$ for the flow. The motion will necessarily be time harmonic and so we introduce $\phi(x, y)$, where

$$\Phi(x, y, t) = \mathrm{Re}\{\phi(x, y)e^{-i\omega t}\}.$$

Then $\phi(x, y)$ is harmonic in the fluid region and satisfies

$$K\phi + \frac{\partial\phi}{\partial y} = 0, \quad K = \omega^2/g$$

on the linearised free surface, $y = 0$.

It is reasonable to assume that part of the incident wave energy will be reflected back to $x = -\infty$, and part will be transmitted through the n^{th} gap towards $x = +\infty$. Thus we shall assume that $\phi \sim \phi_0[\exp(+iKx - Ky) + R\exp(-iKx - Ky)]$ as $x \to -\infty$. and that $\phi \sim \phi_0 T\exp(+iKx - Ky)$ as $x \to +\infty$.

The outer solution

Consider the gap L_n: $|y - h_n| \leqq a_n$, $|x - b_n| \leqq c_n$. Since both $2a_n/h_n$ and $2c_n/b_n$ are assumed small compared to unity, to an observer who is distance $O(h_n)$ from L_n, the barrier will appear to be positioned on the line $x = b_n$ and the flow through L_n will be indistinguishable from an oscillating line source at (b_n, h_n).

Thus for $x > b_n$, and for distances $O(h_n)$ from L_n, we assume

$$\phi(x, y) = m_n G(x, y; b_n, h_n) \tag{2.1}$$

where $G(x, y; b_r, h_r)$ is the usual infinite depth velocity potential due to a source at (b_r, h_r) beneath a free surface (Wehausen and Laitone, [5]). Clearly, from (2.1) $\partial\phi/\partial x = 0$ on $x = b_n$, $y \neq h_n$.

To an observer in the region $b_{n-1} < x < b_n$, a distance $O\{\min(d_n, h_n, h_{n-1})\}$, (where $d_n = b_n - b_{n-1}$) from either L_n or L_{n-1}, the flow through the gap L_n will appear to be due to an oscillating source positioned at (b_n, h_n), having strength $-m_n$, by continuity, which satisfies the condition of no flow across $x = b_{n-1}$ and $x = b_n$. In a similar manner, the flow through the gap L_{n-1} will appear to be due to an oscillating source at (b_{n-1}, h_{n-1}) which also satisfies the condition of no flow across $x = b_{n-1}, b_n$.

Thus for $b_{n-1} < x < b_n$, away from L_n and L_{n-1}, we assume

$$\phi(x, y) = -m_n H_n^-(x, y) + m_{n-1} H_{n-1}^+(x, y) \tag{2.2}$$

where m_n and m_{n-1} are unknown source strengths and $H_r^\pm(x, y)$ is the potential due to a line source at (b_r, h_r) satisfying the free-surface condition and also the conditions $\partial H_r^\pm/\partial x = 0$ on $x = b_r, b_{r\pm 1}$. Expressions for $G(x, y; b_r, h_r)$ and $H_r^\pm(x, y)$ are given in the Appendix.

It is clear that we can write down similar outer solutions for $b_{k-1} < x < b_k$, valid away from the gaps L_k, L_{k-1}, for $k = 2, \ldots, n$.

Thus

$$\phi(x, y) = -m_k H_k^-(x, y) + m_{k-1} H_{k-1}^+(x, y), \qquad k = 2, \ldots, n, \tag{2.3}$$

where m_k, $(k = 1, \ldots, n)$, are unknown source strengths to be determined by matching with an inner solution valid in the neighbourhood of the gaps.

Finally, for $x < b_1$, away from L_1, the outer solution is assumed to be

$$\phi(x, y) = -m_1 G(x, y; b_1, h_1) + A\cos K(b_1 - x)\exp(-Ky) \tag{2.4}$$

where a standing wave of arbitrary amplitude, satisfying the no flow condition on $x = b_1$, is included so that the requirement of an incident wave and a reflected wave at $x = -\infty$ is satisfied.

Now $G(x, y; b_r, h_r) \to -2\pi i \exp\{-K(y + h_r) + iK|x - b_r|\}$ as $|x - b_r| \to \infty$, so that as $x \to +\infty$,

$$\phi(x, y) \to -2\pi i m_n \exp\{-K(y + h_n) + iK(x - b_n)\}$$

and as $x \to -\infty$,

$$\begin{aligned}\phi(x, y) \to\ & +2\pi i m_1 \exp\{-K(y + h_1) - iK(x - b_1)\} \\ & + \tfrac{1}{2}A\exp\{iKx - Ky - iKb_1\} + \tfrac{1}{2}A\exp\{-iKx - Ky + iKb_1\}\end{aligned}$$

It follows that

$$R = (1 + 4\pi i(m_1/A)\exp(-Kh_1))\exp(2iKb_1) \tag{2.5}$$

and

$$T = -4\pi i(m_n/A)\exp(-Kh_n - iK(b_n - b_1)). \tag{2.6}$$

The inner solution

Consider the flow in the neighbourhood of the gap L_i, $i = 1, 2, \ldots, n$. To an observer positioned in the gap, the influence of the free surface will be negligible and the upper part of the barrier will appear to extend to infinity. In this inner region, therefore, the solution is required to the problem of potential flow through a finite rectangular aperture of width $2a_i$ in a wall of thickness $2c_i$, the fluid extending to infinity in all directions. The solution to this problem is made unique to within a constant by requiring source-sink type behaviour at large distances either side of L_i.

This problem has been solved by Guiney, Noye and Tuck [3] in the course of determining the wave transmission through a gap in a single barrier. All that is required is the behaviour of the potential at large distances either side of L_i. Thus we find that as $r_i \to \infty$,

$$\phi(x, y) \to m_i \log r_i - \tfrac{1}{2}m_i \log(k_i\delta_i^2) + C_i \text{ for } x > b_i \tag{2.7}$$

whilst

$$\phi(x, y) \to -m_i \log r_i + \tfrac{1}{2}m_i \log(k_i\delta_i^2) + C_i \text{ for } x < b_i. \tag{2.8}$$

Here $r_i^2 = (x - b_i)^2 + (y - h_i)^2$, $i = 1, 2, \ldots, n$,

$$\delta_i = a_i/(2E(k_i) - k_i'^2K(k_i))$$

and $K(k_i)$, $E(k_i)$ are complete elliptic integrals of the first and second kind respectively. Also, k_i is the positive root of

$$c_i/a_i = K'k_i'^2 - 2K' + 2E')/2(Kk_i'^2 - 2E)$$

where $K' = K(k_i')$, $E' = E(k_i')$ and $k_i' = (1 - k_i^2)^{\frac{1}{2}}$. Details of this derivation and the full solution to the inner problem are given in Guiney, Noye and Tuck [3].

The equations (2.7), (2.8) carry into the outer region information about the geometry of the inner region via the constants k_i, δ_i and C_i. We need now to consider the behaviour of the outer solution in the neighbourhood of the gaps.

We find from (2.3) that, as $r_i \to 0$, for $x > b_i$,

$$\phi \to m_i \log r_i + m_i\beta_i^+ - m_{i+1}\gamma_{i+1}^-, \qquad i = 1, 2, \ldots, n-1, \tag{2.9a}$$

and for $x < b_i$,

$$\phi \to -m_i \log r_i - m_i\beta_i^- + m_{i+1}\gamma_{i-1}^+, \quad i = 2, 3, \ldots, n. \tag{2.9b}$$

Also, from (2.4) as $r_1 \to 0$, for $x < b_1$

$$\phi \to -m_1 \log r_1 - m_1\alpha_1 + A\exp(-Kh_1) \tag{2.9c}$$

and from (2.2) as $r_n \to 0$, $x > b_n$

$$\phi \to m_n \log r_n + m_n\alpha_n. \tag{2.9d}$$

Expressions for the constants $\beta_i^\pm$, $\gamma_i^\pm$, and α_i are given in the Appendix.

The expressions (2.9) carry information about the wave-like nature of the flow from the outer region into the inner region. All that remains is to match the expansions (2.9) with (2.7) and (2.8).

We obtain

$$-m_i\beta_i^- + m_{i-1}\gamma_{i-1}^+ = \tfrac{1}{2}m_i\log(k_i\delta_i^2) + C_i, \quad i = 2, \dots, n,$$
$$m_i\beta_i^+ - m_{i+1}\gamma_{i+1}^- = -\tfrac{1}{2}m_i\log(k_i\delta_i^2) + C_i, \quad i = 1, 2, \dots, n-1,$$
$$-m_1\alpha_1 + A\exp(-Kh_1) = \tfrac{1}{2}m_1\log(k_1\delta_1^2) + C_1,$$
$$m_n\alpha_n = -\tfrac{1}{2}m_n\log(k_n\delta_n^2) + C_n.$$

Elimination of C_i, $i = 1, 2, \dots, n$, gives

$$\left.\begin{aligned} &m_i(\beta_i^+ + \beta_i^- + \log(k_i\delta_i^2)) = m_{i-1}\gamma_i + m_{i+1}\gamma_i^+, \quad i = 2, \dots, n-1\\ &m_1(\alpha_1 + \beta_1^+ + \log(k_1\delta_1^2)) = m_2\gamma_1^+ + A\exp(-Kh_1)\\ &m_n(\alpha_n + \beta_n^- + \log(k_n\delta_n^2)) = m_{n-1}\gamma_n^- \end{aligned}\right\} \tag{2.10}$$

where the identity $\gamma_i^+ = \gamma_{i+1}^-$, $i = 1, \dots, n-1$ proved in the Appendix, has been used. It is convenient at this stage to introduce the real constants p_1, p_n, $q_i^\pm$ and l_i, $i = 1, \dots, n$. Thus we define

$$q_i^\pm = \gamma_i^\pm/(2\pi\exp(-2Kh_i)),$$
$$2l_i = \{\beta_i^+ + \beta_i^- + \log(k_i\delta_i^2)\}/(2\pi\exp(-2Kh_i)),$$
$$p_1 = i + \{\alpha_1 + \beta_1^+ + \log(k_1\delta_1^2)\}/(2\pi\exp(-2Kh_1)),$$
$$p_n = i + \{\alpha_n + \beta_n^- + \log(k_n\delta_n^2)\}/(2\pi\exp(-2Kh_n)).$$

In terms of these new constants equations (2.10) become

$$2l_im_i = m_{i-1}q_i^- + m_{i+1}q_i^+, \quad i = 2, \dots, n-1, \tag{2.11}$$
$$m_1(p_1 - i) = m_2q_1^+ + A/2\pi\exp(-Kh_1), \tag{2.12}$$
$$m_n(p_n - i) = m_{n-1}q_n^-. \tag{2.13}$$

The difference equation (2.11), together with (2.13), enables m_i, $i = 1, 2, \dots, n-1$ to be expressed in terms of m_n, and (2.12) together with (2.5) and (2.6) enable R and T to be written

$$R\exp(-2iKb_1) = \{m_1(p_1 + i) - m_2q_1^+\}/\{m_1(p_1 - i) - m_2q_1^+\} \tag{2.14}$$

and

$$T\exp(-iK(b_1 - b_n)) = -2im_n\exp\{K(h_1 - h_n)\}/\{m_1(p_1 - i) - m_2q_1^+\} \tag{2.15}$$

It is not obvious from these expressions for R and T that $|R|^2 + |T|^2 = 1$ as might be expected from considerations of energy conservation, or by an elementary application of Green's theorem. However it can be shown, after some algebra, that

$$1 - |R|^2 = 4q_1^+\,\mathrm{Im}(\overline{m}_1m_2)/|m_1(p_1 - i) - m_2q_1^+|^2.$$

If we multiply (2.11) by $\overline{m}_i$, (2.13) by $\overline{m}_n$, and take the imaginary parts, we obtain

$$\mathrm{Im}(\overline{m}_{i-1}m_i)q_i^- = \mathrm{Im}(\overline{m}_im_{i+1})q_i^+, \quad i = 2, \dots, n-1,$$
$$\mathrm{Im}(\overline{m}_{n-1}m_n)q_n^- = |m_n|^2.$$

Hence

$$q_1^+ \operatorname{Im}(\overline{m}_1 m_2) = |m_n|^2 \prod_{i=1}^{n-1} (q_i^+/q_{i+1}^-)$$

$$= |m_n|^2 \exp 2K(h_1 - h_n)$$

since from (A.7)

$$q_i^+ \exp(2Kh_{i+1}) = q_{i+1}^- \exp(2Kh_i), \qquad i = 1, 2, \ldots, n-1,$$

and so $|R|^2 + |T|^2 = 1$.

Special cases

If $n = 2$, corresponding to two barriers, then $m_1 = (p_2 - i)m_2/q_2^-$ and we obtain

$$R \exp(-2iKb_1) = \{(p_1 + i)(p_2 - i) - q_1^+ q_2^-\}/\{(p_1 - i)(p_2 - i) - q_1^+ q_2^-\} \qquad (2.16)$$

and

$$T \exp(iKd_2) = -2iq_2^- \exp(K(h_1 - h_2))/\{(p_1 - i)(p_2 - i) - q_1^+ q_2^-\}. \qquad (2.17)$$

It can be seen from the expression for T that the transmission coefficient vanishes if $q_2^- = 0$, or equivalently if $\gamma_2^- \equiv \gamma_1^+ = 0$. Referring to the Appendix equation (A.7) it follows that $T = 0$ provided $H_1^+(b_2, h_2) \equiv H_2^-(b_1, h_1) = 0$. In other words the potential due to a source at (b_1, h_1) beneath a free surface, and bounded by the rigid walls $x = b_1, b_2$ must vanish at the point (b_2, h_2).

The special case of symmetric gaps in barriers of zero thickness has been studied by the author using an alternative approach [4]. In that case $h_1 = h_2 \equiv h$, $b_1 = b_2 \equiv b$ so that the condition $T = 0$ is satisfied if $\gamma = 0$ where γ is given by equations (A.10). This condition agrees with that obtained in [4], where it was also shown that the equation $\gamma = 0$ has an infinity of solutions. The solutions occur, however, for values of the parameters of the problem for which the transmission coefficient is very small anyway and they are of little physical interest. The expressions for T and R can also be shown to agree with those obtained in [4] where curves showing the variation of T with Kh for different values of $2b/h$ are given. For $n = 3$ we obtain for T the expression

$$T e^{iK(b_1 - b_3)} = -2iq_2^- q_3^- e^{K(h_1 - h_3)}/[\{2l_2(p_3 - i) - q_3^- q_2^+\}(p_1 - i) - (p_3 - i)q_2^- q_1^+] \quad (2.18)$$

and it is clear that for $n > 3$ the expressions for R and T become more and more unwieldy.

It is noticeable from (2.18) that $T = 0$ if either q_2^- or q_3^- vanishes, so that for three barriers an additional set of transmission-free configurations is possible, corresponding to the potential due to a source at (b_2, h_2) bounded by the rigid walls $x = b_2, b_3$, vanishing at (b_3, h_3).

It is possible to obtain relatively simple expressions for R and T for the special case of n equally-spaced barriers with symmetric gaps. In this case the solution to the difference equation can be expressed in closed form for arbitrary n. Let the width of each barrier be $2c$, the size of each gap be $2a$, the depth of each gap be h and let the barriers be a distance d from each other. Then for $i = 1, 2, \ldots, n$, we have $a_i = a$, $c_i = c$, $h_i = h$, $d_i = d$, $b_i = ib$, $k_i = k$, say, and $\delta_i = \delta$, say. Then $\alpha_i \equiv \alpha$, $\beta_i^\pm \equiv \beta$, $\gamma_i^\pm \equiv \gamma$, $q_i^\pm \equiv q$, $l_i \equiv l$ and $p_i \equiv p$ for all appropriate values of i.

The difference equation now becomes

$$2lm_i = q(m_{i-1} + m_{i+1}), \quad i = 2, \ldots, n-1, \tag{2.19}$$

with

$$m_n(p - i) = m_{n-1}q \tag{2.20}$$

and

$$m_1(p - i) = m_2 q + (A \exp Kh)/2\pi. \tag{2.21}$$

The equation (2.19) may be solved by standard methods. Thus, let

$$|l/q| = \begin{cases} \cosh\theta, & |l/q| \geqq 1, \\ \cos\theta, & |l/q| \leqq 1. \end{cases} \tag{2.22}$$

Then it can be shown that the solution of (2.19) which satisfies (2.20) is

$$m_i = m_n\{(p - i)s_{n-i} - qs_{n-i-1}\}/qs_1, \quad i = 1, 2, \ldots, n,$$

where

$$s_k = \begin{cases} \sinh k\theta, & l/q \geqq 1, \\ \sin k\theta, & 0 \leqq l/q \leqq 1, \\ (-1)^k \sinh k\theta, & l/q \leqq -1, \\ (-1)^k \sin k\theta, & 0 \geqq l/q \geqq -1. \end{cases}$$

It follows from (2.5) and (2.6) that

$$T\exp\{iK(n-1)b\} = -2iqs_1/\{(p-i)^2 s_{n-1} - 2(p-i)qs_{n-2} + q^2 s_{n-3}\} \tag{2.23}$$

whilst

$$R\exp(-2iKb) = $$
$$= \{(p^2+1)s_{n-1} - 2pqs_{n-2} + q^2 s_{n-3}\}/\{(p-i)^2 s_{n-1} - 2(p-i)qs_{n-2} + q^2 s_{n-3}\}. \tag{2.24}$$

Expressions (2.23) and (2.24) embody the main results of this paper. The transmission and reflection properties of n symmetrical barriers each containing a small gap are expressed in terms of three constants p, q and, through (2.22), l. It is seen from the Appendix equation (A.11) to (A.13) that p, q and l can be written in terms of tabulated functions and rapidly convergent infinite series which may easily be computed.

Appendix

The velocity potential $G(x, y; b_i, h_i)$ due to a line source at (b_i, h_i) beneath a free surface is well known (Wehausen and Laitone [5] and may be written

$$G(x, y; b_i, h_i) = -2\pi i \exp\{-K(y + h_i) + iK|x - b_i|\} + g(x, y; b_i, h_i)$$

where

$$g(x, y; b_i, h_i) =$$

$$= \log(r/r') - 2\int_0^\infty \frac{\{k\cos k(y+h_i) - K\sin k(y+h_i)\}\,e^{-k|x-b_i|}}{k^2+K^2}\,dk \qquad (A.1)$$

$$= -2\int_0^\infty \frac{(k\cos ky - K\sin ky)(k\cos kh_i - K\sin kh_i)e^{-k|x-b_i|}}{k(k^2+K^2)}\,dk \qquad (A.2)$$

and where $r_i = \{(x-b_i)^2 + (y-h_i)^2\}^{\frac{1}{2}}$, $r_i' = \{(x-b_i)^2 + (y+h_i)^2\}^{\frac{1}{2}}$.

As $r_i \to 0$, from (A.1)

$$G(x, y; b_i, h_i) = \log r_i + \alpha_i + o(1)$$

where

$$\alpha_i = -\log 2h_i + 2(\bar{E}i(2Kh_i) - \pi i)\exp(-2Kh_i) \qquad (A.3)$$

where

$$\bar{E}i(x) = \int_{-\infty}^{x} t^{-1}e^t\,dt.$$

We define $H_i^{\pm}(x, y)$ to be the velocity potential due to a line source at (b_i, h_i) under a free surface and bounded by the rigid walls at $x = b_i, b_{i\pm1}$. Then $\partial H_i^{\pm}/\partial x = 0$ on $x = b_i, b_{i\pm1}$ and $H_i^+(x, y)$ is defined for $y \geqq 0$, $b_i \leqq x \leqq b_{i+1}$, $i = 1, 2, \ldots, n-1$ whilst $H_i^-(x, y)$ is defined for $y \geqq 0$, $b_{i-1} \leqq x \leqq b_i$, $i = 2, 3, \ldots, n$.

It follows, by using (A.1) and (A.2) and the method of images, that

$$H_i^+(x, y) = G(x, y; b_i, h_i) + 2\pi e^{-K(y+h_i)+iKd_i}\cos K(x-b_i)/\sin Kd_i$$

$$-2\int_0^\infty \frac{e^{-kd_i}\cosh k(x-b_i)(k\cos ky - K\sin ky)(k\cos kh_i - K\sin kh_i)}{k\sinh kd_i(k^2+K^2)}\,dk \qquad (A.4)$$

$$= 2\pi e^{-K(y+h_i)}\cos K(x-b_{i+1})/\sin Kd_i$$

$$-2\int_0^\infty \frac{(k\cos ky - K\sin ky)(k\cos kh_i - K\sin kh_i)\cosh k(x-b_{i+1})}{k\sinh kd_i(k^2+K^2)}\,dk. \qquad (A.5)$$

Here $d_i = b_{i+1} - b_i$. The function $H_i^-(x, y)$ is the same as (A.4) with d_i replaced by d_{i-1} or (A.5) with d_i replaced by d_{i-1} and b_{i+1} replaced by b_{i-1}.

As $r_i \to 0$,

$$H_i^+(x, y) = \log r_i + \beta_i^+ + o(1)$$

where

$$\beta_i^+ = \alpha_i - 2\int_0^\infty \frac{e^{-kd_i}(k\cos kh_i - K\sin kh_i)^2}{k\sinh kd_i(k^2+K^2)}\,dk + 2\pi e^{-2Kh_i+iKd_i}\operatorname{cosec} Kd_i,$$

$$i = 1, 2, \ldots, n-1.$$

Similarly, $H_i^-(x, y) = \log r_i + \beta_i^- + o(1)$ as $r_i \to 0$ where β_i^- is β_i^+ with d_i replaced by d_{i-1} for $i = 2, 3, \ldots, n$. Notice that β_i is real.

Finally, we define $\gamma_i^\pm \equiv H_i^\pm(b_{i\pm 1}, h_{i\pm 1})$, so that from (A.5)

$$\gamma_i^+ = 2\pi \exp\{-K(h_i + h_{i+1})\} \operatorname{cosec} Kd_i - 2\int_0^\infty \frac{(k\cos kh_{i+1} - K\sin kh_{i+1})(k\cos kh_i - K\sin kh_i)}{k\sin kd_i(k^2+K^2)}\, dk \tag{A.6}$$

and also

$$\gamma_i^+ = \gamma_{i+1}^-, \; i = 1, 2, \ldots, n-1, \tag{A.7}$$

thus verifying the reciprocity relation for symmetric Green's functions.

For the special case of n identical equally spaced barriers and gaps, $h_i = h$, $a_i = a$, $d_i = b$, for $i = 1, 2, \ldots, n$, and so $\beta_i^+ = \beta_i^- \equiv \beta$, independent of i, where

$$\beta = \alpha + 2\pi e^{-2Kh+iKb} \operatorname{cosec} Kb - 2\int_0^\infty \frac{e^{-kb}(k\cos kh - K\sin kh)^2}{k\sin kb(k^2+K^2)}\, dk \tag{A.8}$$

and

$$\alpha \equiv \alpha_i = -\log 2h + 2e^{-2Kh}(\bar{E}i(2Kh) - \pi i). \tag{A.9}$$

Also $\gamma_i^+ = \gamma_i^- \equiv \gamma$, independent of i, where

$$\gamma = 2\pi e^{-2Kh} \operatorname{cosec} Kb - 2\int_0^\infty \frac{(k\cos kh - K\sin kh)^2}{k\sinh kb(k^2+K^2)}\, dk. \tag{A.10}$$

This expression for γ also occurs in Evans [4] where it was shown, using contour integration, that it could be replaced by a simpler expression which did not involve infinite integrals. The expression for β can be treated similarly so that, with $q_i^\pm \equiv q$,

$$\begin{aligned} q &= \gamma/2\pi \exp(-2Kh) \\ &= \left\{\frac{1}{2Kb} - \frac{1}{2\pi}\log\cosh\left(\frac{\pi h}{b}\right) - \sum_{n=1}^\infty \frac{(-1)^n \exp(-2n\pi h/b)}{n\pi - Kb}\right\}\Big/\exp(-2Kh) \end{aligned} \tag{A.11}$$

and with $l_i \equiv l$,

$$\begin{aligned} l &= (2\beta + \log(k\delta^2))/2\pi \exp(-2Kh) \\ &= \left\{\frac{1}{2Kb} - \frac{1}{2\pi}\log(\sinh(\pi h/b)/(\pi h/b) - \sum_{n=1}^\infty \frac{\exp(-2n\pi h/b)}{n\pi - Kb} \right.\\ &\qquad \left. - \frac{1}{2\pi}\log(2h/k^{\frac{1}{2}}\delta)\right\}/\exp(-2Kh). \end{aligned} \tag{A.12}$$

Finally, with $p_1 = p_n \equiv p$,

$$\begin{aligned} p &= i + \{\alpha + \beta + \log(k\delta^2)\}/2\pi\exp(-2Kh) \\ &= l + \frac{1}{\pi}\bar{E}i(2Kh) - \frac{1}{2\pi}\exp(2Kh).\log(2h/k^{\frac{1}{2}}\delta). \end{aligned} \tag{A.13}$$

Notice that p, q, l are real.

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